JOURNAL OF FINANCIAL AND QUANTITATIVE ANALYSIS
Volume XV, No. 2, June 1980

CAPITAL ASSET PRICING WITH PROPORTIONAL TRANSACTION COSTS

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I. Introduction and Conclusions

The implications for portfolio behavior and asset prices of transaction costs are central to the analysis of numerous issues in economics. For example, questions involving the demand for the financial contracts issued by financial intermediaries are intimately tied to the existence of transaction costs.[1] Thus the analysis of questions involving the nature of the demand for mutual fund shares, insurance contracts, mortgage loans, etc., and the form those contracts take require the explicit inclusion of transaction costs.

The equilibrium structure of asset prices in perfect markets has been examined by Treynor [13], Sharpe [12], Lintner [5], and Mossin [11]. Subsequently, Black [2] produced a variant of this model where there are restrictions on the holding of the risk-free asset; and Brennan [3] allowed different borrowing and lending rates on the risk-free asset. Mayers [8,9] extended the basic model to a situation where the set of assets can be divided into two sets: one containing costlessly marketable assets, and the other containing non-marketable assets (assets for which the cost of trading is infinite). Finally Magill and Constantinedes [6] and Magill [7] analyzed portfolio selection in a continuous-time diffusion model, where there are costs of trading in securities. These last two papers do not attempt an analysis of the structure of asset prices, but concentrate on the analysis of efficient portfolio regions.

In this paper we provide an examination of individual portfolio selection and equilibrium asset pricing with proportional transaction costs. We model transaction costs as a bid-ask differential and derive an equilibrium

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[1] Given that a financial intermediary merely sells a repackaged portfolio of existing securities, then that intermediary can exist in a zero transaction cost equilibrium only if it can be operated at zero cost--and social welfare would be unaffected by the presence, or absence, of that intermediary. See Benston and Smith [1], and also Magill [6,7].

characterization of the structure of bid and asked prices. Our model is based on the single-period Sharpe-Lintner-Mossin model and effectively generalizes the earlier work of Black, Brennan and Mayers. The major findings are:

(1) Individuals do not hold the same portfolios of risky assets: an individual tends to hold his endowment portfolio, and will only change his portfolio holdings if the benefits in terms of increased expected future consumption, or reduction of the variance of future consumption, are greater than the costs in terms of foregone consumption from the payment of transaction costs.

(2) Every individual's efficient portfolio can be obtained as a linear combination of a common portfolio, and a personalized portfolio, drawn from a common set of portfolios. This efficient portfolio set can be generated from market data, and includes the familiar zero-transaction cost efficient set as a special case.

(3) There exists a "shadow price" asset pricing equation that depends upon individual utility information and endowments. The shadow price is unobservable, but must lie between the buying and the selling prices.

The plan of the paper is as follows: in Section II we set out the individual's consumption-portfolio decision; in Section III there is an analysis of the efficient portfolio set and market equilibrium; in Section IV there is an analysis of various subcases of the general model; and in Section V there are some concluding comments of a general nature.

II. The Individual's Consumption-Portfolio Decision

We assume that consumers' preferences are a positive function of current consumption, c_i, a positive function of expected "end-of-period" wealth, w_i, and a negative function of the variance of "end-of-period" wealth, v_i:

$$U^i = U^i(c_i, w_i, v_i), \tag{1}$$

where

$$\frac{\partial U^i}{\partial c_i} \equiv U_c^i > 0, \ \frac{\partial U^i}{\partial w_i} \equiv U_w^i > 0, \ \frac{\partial U^i}{\partial v_i} \equiv U_v^i < 0.$$

Consumer i's expected "end-of-period" wealth is the sum of his fractional ownership of the total expected "end-of-period" values of the firms in the market:

$$w_i \equiv \underline{X}_i' \underline{R}\ , \tag{2}$$

where

$\underline{X}_i$ is a column vector $[X_{i1}, X_{i2}, \ldots, X_{iN}]'$, where X_{ij} is the fraction of firm j's shares held by consumer i;

$\underline{R}$ is a column vector $[R_1, R_2, \ldots, R_N]'$, where R_j is the expected end of period total dollar value (price plus dividend) of firm j's shares.

The variance of consumer i's "end-of-period" wealth is

(3) $$v_i = \underline{X}_i' \, \Sigma \, \underline{X}_i \; ,$$

where

Σ is the variance-covariance matrix of the vector of random variables $[\tilde{R}_1, \tilde{R}_2, \ldots, \tilde{R}_N]'$.

Consumer i's holding of assets is the sum of his initial endowment (if any) plus what he purchases minus what he sells:

(4) $$\underline{X}_i = \underline{X}_j^* + \underline{\Delta}_i^B - \underline{\Delta}_i^S \; ,$$

where

$\underline{X}_i^*$ is a column vector $[X_{i1}^*, X_{i2}^*, \ldots, X_{iN}^*]'$, where X_{ij}^* is the fraction of firm j's shares in consumer i's endowment;

$\underline{\Delta}_i^B$ is a column vector $[\Delta_{i1}^B, \Delta_{i2}^B, \ldots, \Delta_{iN}^B]'$, where Δ_{ij}^B is the fraction of firm j's shares purchased by consumer i;

$\underline{\Delta}_i^S$ is a column vector $[\Delta_{i1}^S, \Delta_{i2}^S, \ldots, \Delta_{iN}^S]'$, where Δ_{ij}^S is the fraction of firm j's shares sold by consumer i.

Individual i's current consumption (in dollars) is the sum of his initial endowment of dollars, c_i^*, plus the value of any assets sold, minus the value of any assets bought:

(5) $$c_i = c_i^* + \underline{\Delta}_i^{S'} \underline{V}^S - \underline{\Delta}_i^{B'} \, \underline{V}^B \; ,$$

where

$\underline{V}^S$ is a column vector $[V_1^S, V_2^S, \ldots, V_N^S]'$, where V_j^S is the current market value of firm j, evaluated using the selling (bid) price;

$\underline{V}^B$ is a column vector $[V_1^B, V_2^B, \ldots, V_N^B]'$, where V_j^B is the current market value of firm j, evaluated using the buying (asked) price.

Finally, we must restrict agents in the economy from buying at bid prices or selling at asked prices,[2]

$$\underline{\Delta}_i^B,\ \underline{\Delta}_i^S \geq \underline{0}. \tag{6}$$

Consumer i's problem can be stated as maximize his utility function in (1) subject to the set of constraints, (2), (3), (4), (5), and (6). To solve this problem, we use standard Kuhn-Tucker techniques; we form a Lagrangian, and substitute (2), (3), (4), and (5) into (1);

$$\Lambda = U^i\left[(c_i^* + \underline{\Delta}_i^{S'}\underline{V}^S - \underline{\Delta}_i^{B'}\underline{V}^B),\ (\underline{X}_i^* + \underline{\Delta}_i^B - \underline{\Delta}_i^S)'\underline{R},\ (\underline{X}_i^* + \underline{\Delta}_i^B - \underline{\Delta}_i^S)' \ddagger (\underline{X}_i^* + \underline{\Delta}_i^B - \underline{\Delta}_i^S)\right] + \underline{\Delta}_i^{B'}\underline{\lambda}_i^B + \underline{\Delta}_i^S\underline{\lambda}_i^{S'}. \tag{7}$$

The first order conditions for individual i with respect to the optimal fraction of shares bought and sold in each firm are,

$$\frac{\partial\Lambda}{\partial\Delta_i^S} = U_c^i\underline{V}^S - U_w^i\underline{R} - 2U_v^i \ddagger (\underline{X}_i^* + \underline{\Delta}_i^B - \underline{\Delta}_i^S) + \underline{\lambda}_i^S = \underline{0}, \tag{8a}$$

$$\frac{\partial\Lambda}{\partial\Delta_i^B} = -U_c^i\underline{V}^B + U_w^i\underline{R} + 2U_v^i \ddagger (\underline{X}_i^* + \underline{\Delta}_i^B - \underline{\Delta}_i^S) + \underline{\lambda}_i^B = \underline{0}. \tag{8b}$$

And from the Kuhn-Tucker Theorem, we know that if the fraction of shares bought or sold in a firm is positive, the Kuhn-Tucker multiplier will be zero:

$$\underline{\Delta}_i^{B'}\underline{\lambda}_i^B = 0,\ \underline{\Delta}_i^B \geq \underline{0},\ \underline{\lambda}_i^B \geq \underline{0}. \tag{9a}$$

$$\underline{\Delta}_i^{S'}\underline{\lambda}_i^S = 0,\ \underline{\Delta}_i^S \geq \underline{0},\ \underline{\lambda}_i^S \geq \underline{0}. \tag{9b}$$

If we sum (8a) and (8b), we find,

$$U_c^i(\underline{V}^B - \underline{V}^S) = \underline{\lambda}_i^S + \underline{\lambda}_i^B \geq \underline{0}. \tag{10}$$

Since increases in current consumption increase utility $[U_c^i > 0]$, the asked price must be no less than the bid price $[(\underline{V}^B - \underline{V}^S) \geq \underline{0}]$, if the solution is to be

[2] The model in this paper is not really closed because it omits any discussion of the transaction cost technology that generates the differing buying and selling prices. For a more general model, see Milne [10].

bounded. Hence, there are four cases to consider: {1} if λ^S_{ik}, $\lambda^B_{ik} > 0$, then from (9) $\Delta^S_{ik} = \Delta^B_{ik} = 0$; {2} if $\lambda^S_{ik} > 0$ and $\lambda^B_{ik} = 0$, then $\Delta^S_{ik} = 0$ and $\Delta^B_{ik} \geq 0$; {3} if $\lambda^S_{ik} = 0$ and $\lambda^B_{ik} > 0$, then $\Delta^S_{ik} \geq 0$ and $\Delta^B_{ik} = 0$; and {4} if $\lambda^S_{ik} = \lambda^B_{ik} = 0$, then $\Delta^S_{ik} \geq 0$ and $\Delta^B_{ik} \geq 0$. Note that from (10), cases {1}, {2}, and {3} occur when the bid price exceeds the asked price; while case {4} occurs when the two prices are equal.

To derive a more intuitive understanding of these restrictions, again consider equation (8). Define $\underline{U}^i_\Delta$ as the vector of marginal utilities from changes in the vector of security holdings, both through the effect on portfolio expected payout and through the effect on the variance of the project:

(11) $$\underline{U}^i_\Delta \equiv U^i_w \, \underline{R} + 2\, U^i_v \, \underline{\Sigma} \, (\underline{X}^*_i + \underline{\Delta}^B_i - \underline{\Delta}^S_i).$$

Thus U^i_Δ is a column vector $[U^i_{\Delta 1}, U^i_{\Delta 2}, \ldots, U^i_{\Delta N}]'$, where $U^i_{\Delta j}$ is individual i's marginal utility from the change in the holding of security j, both through the effect on portfolio expected payout and through the effect on portfolio variance.

Substituting this definition of marginal utilities in (11) into the first order conditions specified in (8) yields:

(12a) $$U^i_c \underline{V}^S - \underline{U}^I_\Delta + \underline{\lambda}^S_i = \underline{0},$$

(12b) $$U^i_c \underline{V}^B - \underline{U}^i_\Delta - \underline{\lambda}^B_i = \underline{0}.$$

Equation (12) can be rearranged to yield

(13) $$\underline{U}^i_\Delta \,/\, U^i_c = \underline{V}^S + (\underline{\lambda}^S_i \,/\, U^i_c) = \underline{V}^B - (\underline{\lambda}^B_i \,/\, U^i_c).$$

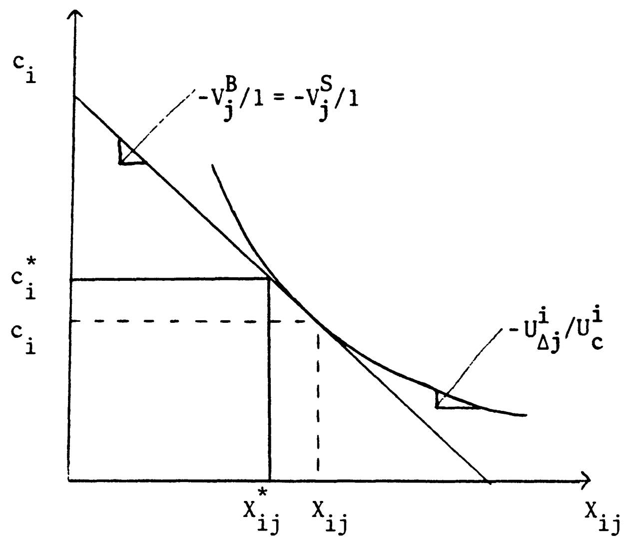
By assumption, buying prices exceed selling prices $(\underline{V}^B \geq \underline{V}^S)$ and the marginal utility of consumption is positive $(U^i_c > 0)$, while from (9) we know that the Kuhn-Tucker multipliers are nonnegative, $\underline{\lambda}^S_i, \underline{\lambda}^B_i \geq \underline{0}$; thus (13) can be expressed as

(14) $$\underline{V}^B \geq \underline{U}^i_\Delta \,/\, U^i_c \geq \underline{V}^S.$$

Consider four cases: {1} $V^B_j = U^i_{\Delta j} / U^i_c = V^S_j$; {2} $V^B_j > U^i_{\Delta j} / U^i_c = V^S_j$; {3} $V^B_j = U^i_{\Delta j} / U^i_c > V^S_j$; and {4} $V^B_j > U^i_{\Delta j} / U^i_c > V^S_j$.

In case {1}, bid and asked prices are equal. Graphically, the individual's consumption-portfolio decision can be represented as in Figure 1.

FIGURE 1



INDIVIDUAL i's CONSUMPTION-PORTFOLIO DECISION where c_i is the consumption of individual i in the first period and X_{ij} is the fraction of firm j owned by individual i. X_{ij}^* and c_i^* are initial endowments. The slope of the indifference curve is $-U^i_{\Delta j}/U^i_c$. The slope of the opportunity set is $-V_j^B/1 = -V_j^S/1$.

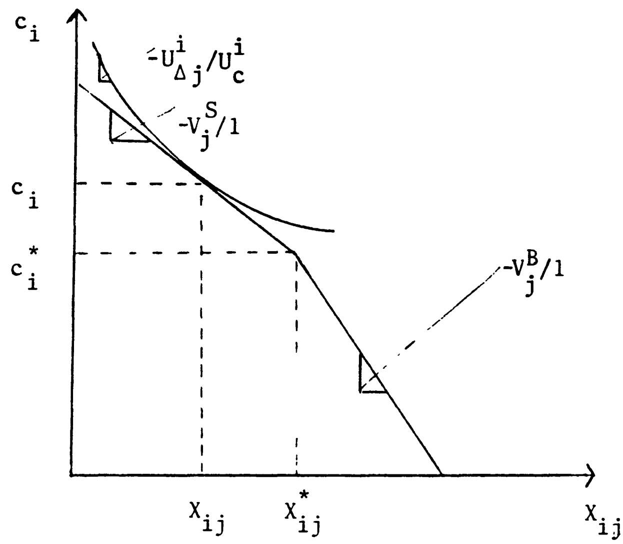
This is the standard case; the individual equates his marginal rate of substitution between consumption and X_j (MRS_{cj}) with the ratio of the prices (note, since current consumption is in dollars, $P_c = 1$).

In case {2}, the bid price is less than the asked price and in equilibrium the individual's marginal rate of substitution between consumption and X_j equals the ratio of relative prices using bid prices ($MRS_{cj} = V_j^S/1$). Figure 2 illustrates this case. The individual sells $\Delta_{ij}^S \equiv X_{ij}^* - X_{ij}$, increasing consumption by $-\Delta_{ij}^S V_j^S \equiv c_i^* - c_i$.

Cases {3} and {4} can also be described using Figure 2. In case {3}, the individual sells part of his endowment of the consumption good and purchases additional shares of firm j. The tangency between indifference curve and opportunity set occurs to the right of the endowment point. In case {4} there will be no trades, and there is no tangency between opportunity set and indifference curve.[3] The individual's highest attainable indifference curve touches the opportunity set at the endowment point.

[3]In cases {1}, {2}, and {3}, there will also be no trades if the tangency point occurs at the endowment point.

FIGURE 2



INDIVIDUAL i's CONSUMPTION-PORTFOLIO DECISION WHERE BID AND ASKED PRICES DIFFER where c_i is individual i's first period consumption and X_{ij} is the fraction of firm j owned by individual i. X^*_{ij} and c^*_{ij} are initial endowments. The slope of the indifference curve is $-U^i_{\Delta j}/U^i_c$. The slope of the opportunity set above the endowment point (for sales of X_{ij}) is $-V^S_j/1$; below the endowment point (for purchases of X_{ij}) it is $-V^B_j/1$.

With positive transaction costs, the individual will tend to hold initial endowments; he will trade only if the marginal gain from transacting exceeds the marginal cost. Figure 2 represents the marginal cost in terms of the slope of the opportunity set; it represents the market terms of trade and the cost of one asset in terms of current consumption opportunities. The indifference curve represents the individual's tradeoff between security holdings (incorporating the effects on the portfolio's expected payoff and variance) and consumption.

III. Market Equilibrium

To consider the demand for assets implied by the first order conditions in (8), we will temporarily assume that there are no riskless assets. Assuming the variance-covariance matrix is of full rank, then its inverse exists. Thus from (8a) we derive the vector of optimal asset holdings by individuals:

$$\underline{X}_i \equiv \underline{X}^*_i + \underline{\Delta}^B_i - \underline{\Delta}^S_i = \Sigma^{-1}[\alpha_i \underline{V}^S - \theta_i \underline{R} + \gamma_i \underline{\lambda}^S_i], \tag{15a}$$

where

$$\alpha_i \equiv U_c^i/2U_v^i > 0,\ \theta_i \equiv U_w^i/2U_v^i < 0,\ \gamma_i \equiv 1/2U_v^i < 0.$$

Equivalently, from (8b) we derive

$$\text{(15b)} \qquad \underline{X}_i \equiv \underline{X}_i^* + \underline{\Delta}_i^B - \underline{\Delta}_i^S = \ddagger^{-1}\ [\alpha_i \underline{V}^B - \theta_i \underline{R} - \gamma_i \underline{\lambda}_i^B].$$

Equating the two expressions in (15a) and (15b) yields:

$$\alpha_i [\underline{V}^B - \delta_i \underline{\lambda}_i^B] = \alpha_i [\underline{V}^S + \delta_i \underline{\lambda}_i^S],$$

where

$$\delta_i \equiv (\gamma_i/\alpha_i) > 0.$$

Now define $V_{ji} \equiv V_j^B - \delta_i \lambda_{ji}^B = V_j^S + \delta_i \lambda_{ji}^S$. Clearly, $V_j^B \geq V_{ji} \geq V_j^S$. Therefore (15a) and (15b) can be combined:

$$\text{(16)} \qquad \underline{X}_i = \ddagger^{-1} [\alpha_i \underline{V}_i - \theta_i \underline{R}]\ ,$$

where

$$\underline{V}_i \equiv [V_{1i}, \ldots, V_{ji}, \ldots, V_{ni}]'.$$

Notice that the vector $\underline{V}_i$ is chosen from a rectangular region

$$\text{(17)} \qquad K \equiv \{\underline{V}_i \varepsilon R^N | V_j^B \geq V_{ij} \geq V_j^S,\ V_j = 1, \ldots, N\}\ ,$$

so that any consumer i chooses a portfolio which is a linear map of a combination of the vector $\underline{R}$ and a vector $\underline{V}_i \varepsilon K$. Therefore, in general, individuals will not hold the market portfolio of risky securities. In the limiting case, with no transaction costs, (i.e., $\underline{V}^B = \underline{V}^S$) the set K reduces to a single vector, $K=\{\underline{V}\}$ and (16) takes the familiar form

$$\text{(18)} \qquad \underline{X}_i = \ddagger^{-1} [a_i \underline{V} - \theta_i \underline{R}].$$

Although the set of efficient portofolios in (16) is much less restrictive than the usual (18), it does imply weak testable restrictions. Now we turn to a characterization of equilibrium asset prices.

If we assume that there are k consumers, market clearing implies that across all individuals, the sum of the fractional ownership of each firm is 1:

(19) $$\sum_{i=1}^{k} \underline{X}_i \equiv \sum_{i=1}^{k} (\underline{X}_i^* + \underline{\Delta}_i^B - \underline{\Delta}_i^S) = \sum_{i=1}^{k} \underline{X}_i^* = \underline{1} \ .$$

Hence this implies that net trades between consumers sum to zero:

(20) $$\sum_{i=1}^{K} (\underline{\Delta}_i^B - \underline{\Delta}_i^S) = \underline{0}.$$

Substituting (16) into (19), and rearranging yields,

(21) $$\sum_i \alpha_i \underline{V}_i = \ddagger \underline{1} + \theta \underline{R}$$

where

$$\theta \equiv \sum_i \theta_i < 0 \ .$$

Now observe that $\alpha_i \underline{V}^S \geq \alpha_i \underline{V}_i \geq \alpha_i \underline{V}^B$, implies

(22) $$\underline{V}^S \sum_i \alpha_i \geq \sum_i \alpha_i \underline{V}_i \geq \underline{V}^B \sum_i \alpha_i \ ,$$

or

(23) $$\underline{V}^S \leq (\sum_i \alpha_i \underline{V}_i)/(\sum_i \alpha_i) \leq \underline{V}^B \ .$$

Defining $\underline{\tilde{V}} \equiv (\sum_i \alpha_i \underline{V}_i)/(\sum_i \alpha_i)$, then using (21)

(24) $$\underline{V}^S \leq \underline{\tilde{V}} = \eta_1 \ddagger \underline{1} + \eta_2 \underline{R} \leq V^B \ ,$$

where

$$\alpha \equiv \sum \alpha_i \ , \ \eta_1 \equiv (\alpha)^{-1} \ , \ \eta_2 \equiv (\theta/\alpha) \ .$$

We now solve (24) for the values of η_1 and η_2 following standard techniques. To eliminate η_1, multiply (24) by $\underline{1}'$,

(25) $$\tilde{V}_M = \eta_1 \sigma_m^2 + \eta_2 \bar{R}_m$$

where

$$\tilde{V}_m \equiv \underline{1}'\underline{\tilde{V}}, \ \sigma_m^2 \equiv \underline{1}' \ddagger \underline{1}, \ \bar{R}_m \equiv \underline{1}'\underline{R} \ .$$

Solve (25) for η_1 and substitute into (24),

(26) $$\underline{V}^S \leq \underline{\tilde{V}} = [(\tilde{V}_m - \eta_2 \bar{R}_m)/ \sigma_m^2] \ \ddagger \underline{1} + \eta_2 \underline{R} \leq \underline{V}^B \ .$$

To eliminate η_2, choose a vector $\underline{\mathcal{Z}}$ such that $\underline{\mathcal{Z}}'\ddagger\underline{1} = 0$.
Multiply (24) by $\underline{\mathcal{Z}}'$,

$$\tilde{V}_{\mathcal{Z}} = \eta_2 \bar{R}_{\mathcal{Z}} \tag{27}$$

where

$$\tilde{V}_{\mathcal{Z}} \equiv \underline{\mathcal{Z}}'\underline{\tilde{V}} \ , \ \bar{R}_{\mathcal{Z}} \equiv \underline{\mathcal{Z}}'\underline{R} \ .$$

Substituting (27) into (26) and simplifying yields:

$$V^S \leq \tilde{V} = [1/(1+\overset{\approx}{r}_{\mathcal{Z}})] \ [\underline{R}+[\tilde{V}_m(1+\overset{\approx}{r}_{\mathcal{Z}})-\bar{R}_m]\beta] \leq V^B \tag{28}$$

where

$$(1+\overset{\approx}{r}_{\mathcal{Z}}) \equiv \bar{R}_{\mathcal{Z}}/\tilde{V}_{\mathcal{Z}}$$

$$\underline{\beta} \equiv (\ddagger\underline{1})/\sigma_m^2 \ .$$

The inequalities in (28) provide (a) a bound on the "shadow value," $\underline{\tilde{V}}$, of the total share values for firms; and (b) an implicit pricing equation for $\underline{\tilde{V}}$ that is analogous to the Black two-factor model. Of course, the pricing formula includes the nonobservable $\underline{\tilde{V}}$ terms, and this precludes any simple, direct empirical tests. In this general case then, individual utility information cannot be removed from the equilibrium "pricing" equation.

IV. Special Cases

It should be obvious to the reader that our model includes, as special cases, {1} the Sharpe-Lintner model, {2} the Black model, {3} the Brennan model, and {4} the Mayers model. For completeness, we will show how each of these models emerges as a special case of our model.

1. The Black Two-Factor Model and Sharpe-Lintner Model

If $V^S = V^B \equiv V$, and there is no riskless asset, the inequality (28) collapses to the total dollar return version of Black's model,

$$\underline{V} = [1/(1+\bar{r}_{\mathcal{Z}})] \ [\underline{R} + [V_m(1+\bar{r}_{\mathcal{Z}})-\bar{R}_m]\beta] \ , \tag{29}$$

where

$$(1+\bar{r}_{\mathcal{Z}}) \equiv (\bar{R}_{\mathcal{Z}}/V_{\mathcal{Z}}) \ .$$

If borrowing and lending at the riskless rate r_f is allowed, then (29) reduces to the familiar total dollar return version of the Sharpe-Lintner capital asset pricing model (see Fama-Miller [4]).

$$(30) \qquad \underline{V} = [1/(1+r_f)]\ [\underline{R} + [V_m(1+r_f)-\bar{R}_m]\underline{\beta}],$$

where

r_f is the risk-free rate of interest, $(1+r_f) \equiv (R_f/V_f)$.

2. Brennan's Model

If all risky assets are traded with zero transaction costs, but the riskless asset does incur transaction costs, then (28) becomes,

$$(31) \qquad \underline{V} = [1/(1+\tilde{r}_f)]\ [\underline{R}+[V_m(1+\tilde{r}_f) - \bar{R}_m]\underline{\beta}] \ ,$$

where

$(1+\tilde{r}_f) \equiv R_f/\tilde{V}_f$; and from (23), $\tilde{V}_f \equiv [1/\sum_i \alpha_i][\sum_i \alpha_i V_{if}]$.

Taking $V_{if} = \phi_i V_f^B + (1-\phi_i)V_f^S$, where $\phi_i \varepsilon [0, 1]$, the pricing equation (31) has the same form as Brennan's equation (in the total dollar return version).

3. Mayers' Model

Mayers divides all assets into marketable and nonmarketable assets. Within the framework of this paper, marketable assets have zero transaction costs associated with their purchase or sale (i.e., buying and selling prices are equal), and nonmarketable assets have infinite transaction costs associated with their purchase or sale. For marketable assets, (28) simplifies considerably. Let the set of assets $\{1,\ldots,N\}$ be partitioned such that assets $m \equiv \{1,\ldots,J\}$ are marketable, and assets $\tilde{m} \equiv \{J+1,\ldots,N\}$ are nonmarketable.

For a marketable asset $j \varepsilon m$, buying and selling prices are equal, $V_j^S = V_j^B$; then assuming a riskless asset, (28) can be written,

$$(32) \qquad V_j = [1/(1+r_f)][R_j+V_m(1+r_f) - \bar{R}_m]\beta_j^* \ ,$$

where

$$(1+r_f) \equiv R_f/V_f \ .$$

$$\beta^*_j \equiv \left[\frac{\text{cov}(R_j,R_m) + \text{cov}(R_j,R_{\tilde{m}})}{\text{var}(R_m) + \text{cov}(R_m,R_{\tilde{m}})}\right],$$

$$\text{cov}(R_j, R_m) \equiv \sum_{i \epsilon m} \text{cov}(R_i, R_j) \ ,$$

$$\text{cov}(R_j, R_{\tilde{m}}) \equiv \sum_{i \epsilon \tilde{m}} \text{cov}(R_i, R_j) \ .$$

Equation (32) is the total dollar return form of Mayers' nomarketable asset model with riskless borrowing and lending. Notice that we do not necessarily require that there be infinite transaction costs on the nommarketable assets--finite transaction costs that preclude trading are sufficient.

V. Concluding Comments

Within this framework, the questions of completeness of markets can be addressed with the addition of supply-side considerations. We can view the difference between buying and selling prices as the compensation received by the market maker on each share traded. Thus the market maker's total compensation is the buying-selling spread times the number of shares traded.

In a competitive market we expect the transaction costs will be such that the market makers earn a normal rate of return. However, for some securities, there will be no spread which allows the market maker to earn a normal rate of return. Those markets will remain closed. This result obtains because if the spread is increased, the number of shares traded is expected to fall; thus an increase in the spread can reduce the total compensation received by market makers.[4]

For other questions involving supply side considerations, the assumption of proportional transaction costs is too restrictive. Although within this framework, there is a specific demand for the contracts sold by financial intermediaries, proportional transaction costs do not provide an appropriate microeconomic basis for questions of the production of these contracts.[5] We think that economies of scale in transactions costs is a necessary condition for the examination of the production of mutual fund shares, for example. Because of the nonconvexities which arise in that problem it is significantly more difficult

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[4]For further discussions along these lines, see Milne [10].

[5]Although a nonzero demand for money is possible within the framework of this model, we consider those implications to be relatively uninteresting. We feel that the demand for money arises from the costs of exchanging assets for consumption goods, given some stochastic expenditure patterns; thus, individuals will choose to hold assets which have low transaction costs associated with conversion to consumption goods. Therefore, we feel that the analysis of the demand for money requires a continuous time (or at least a multiperiod) framework for the essential nature of that asset to be explicitly captured.

than the proportional transaction costs case examined here, and we leave that as a problem for further research.

We expect that the most interesting implications of a model employing less restrictive assumptions about the form of transaction costs would involve the characterization of the individual's portfolio choice rather than the characterization of the equilibrium structure of asset prices. Our results indicate with these simpler assumptions the prospect of testing the implications of this analysis is small because the prices are functions of unobservable utility information.

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